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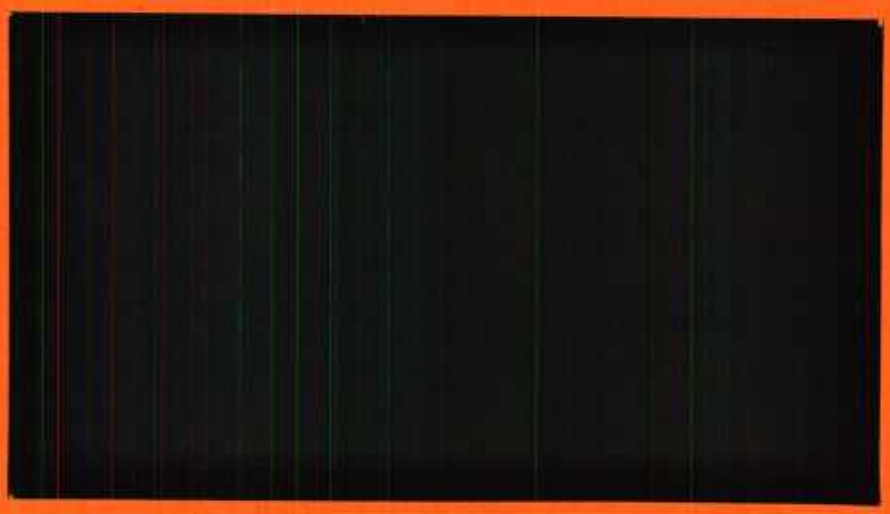

ELECTRICAL ENGINEERING & COMPUTER SCIENCE

STRICT REDUNDANCY SCHEMES FOR NON-SEQUENTIAL DETECTOR RELIABILITY: PART I. ANALYSIS

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ABSTRACT

In this work, we analyze the effects of hardware faults on the performance of computer-implemented signal detectors, as measured by the probability of detection and the probability of false alarm. We derive performance bounds for three implementations: (i) no redundancy, (ii) error masking through strict redundancy, and (iii) error detection and masking through strict redundancy.

I. INTRODUCTION

Given a data sample of fixed size, a non-sequential detector generates a decision variable d_*. If $d_* = 1$, we decide that the signal is present, and if $d_* = 0$ we decide that the signal is absent. Let S be the event that the signal is present and let S^c be the event that the signal is absent. The performance of the detector is generally measured by the probability of detection P_{D*} and the probability of false alarm P_{FA*}, where

$$P_{D*} = P(d_*=1|S)$$

$$P_{FA*} = P(d_*=1|S^c).$$

Existing treatments (see, for example, [1]) of such detection problems implicitly assume that the detector is implemented on hardware that is never faulty. In reality, the computational hardware used to compute the the decision variable is liable to failures. It is therefore reasonable to analyze the effect of possible hardware failures on the detector performance criteria and to ascertain if the possibility of failures can or cannot be neglected.

II. HARDWARE FAULT EFFECTS

Suppose that the detector is implemented on a computing device which calculates a decision variable d_1. The performance measures for

the implemented detector are

$$P_D = P(d_1=1|S)$$

$$P_{FA} = P(d_1=1|S^c).$$

One would like the implemented detector to have the same performance as the theoretical detector. This will obviously be the case if one assumes that the computed decision variable d_1 is always equal to the theoretical decision variable d_*. All existing treatments of the signal detection problem tacitly make this assumption. We will now determine the maximum performance degradation that can occur when such an assumption is not made. Instead, we shall assume that the presence of the signal and the value of the theoretical decision variable d_* are each independent of the correctness of the computations. Precisely, we shall assume the following:

Hypothesis 1: The events S and $(d_*=1)$ are each independent of the event $(d_1=d_*)$.

We can now establish the following theorem. All proofs appear in the appendix.

Theorem 1: If Hypothesis 1 is satisfied, then

$$P_D \geq P(d_1=d_*)P_{D*}$$

$$P_{FA} \leq 1 - P(d_1=d_*)(1 - P_{FA*}).$$

Now if the computing device is non-faulty, the computed decision variable will equal the theoretical decision variable. Therefore, if q is the probability that the computing device is non-faulty, then

$$P(d_1=d_*) \geq q$$

and the following corollary is immediately obtained.

Corollary 1: If Hypothesis 1 is satisfied, then

$$P_D \geq qP_{D*} \tag{1}$$

$$P_{FA} \leq 1 - q(1 - P_{FA*}). \tag{2}$$

Equations (1) and (2) show that the actual performance of the hardware-implemented detector may be quite different from the theoretical performance. For example, assume that a theoretical detector has $P_{FA*} = 10^{-6}$. Even if this detector is implemented on a computing device with q = 0.9999, P_{FA} may be as high as 10^{-4}, two orders of magnitude larger than P_{FA*}.

III. PERFORMANCE USING ERROR MASKING

One way to possibly minimize detector performance degradation in the presence of hardware faults is to use an error masking scheme based on hardware redundancy. To implement such a scheme we replicate the original computing device α-1 times, send the original data sample to each device, and then send all the device outputs $d_1, d_2, \ldots, d_\alpha$ to an

error masker which chooses an index i(m) in the set {1, 2,..., α}. The masker makes its choice by first partitioning the set of device outputs into two blocks such that all the outputs in a given block are identical, and then by choosing the index i(m) at random from among the indices of the outputs in the block(s) of maximal cardinality. The output of the error masker is then $d_{i(m)}$. The decision concerning the presence or absence of the signal is made using the variable $d_{i(m)}$.

When this error masking scheme is used, the performance measures of the implemented signal detector are the probability of detection $P_{D,I}(\alpha)$ and the probability of false alarm $P_{FA,I}(\alpha)$, which are now defined as follows:

$$P_{D,I}(\alpha) = P(d_{i(m)}=1 \mid S)$$

$$P_{FA,I}(\alpha) = P(d_{i(m)}=1 \mid S^c).$$

The classical approach [2] to the analysis of such an error masking scheme is based on assumptions which ensure that $d_{i(m)}$ is always equal to d_*, in which case $P_{D,I}(\alpha) = P_{D*}$ and $P_{FA,I}(\alpha) = P_{FA*}$. For example, one might put an upper bound on the number of computing devices that can produce incorrect results. Since we feel that such assumptions are physically unreasonable, we shall instead assume that the presence of the signal and the value of the theoretical decision variable d_* are each independent of the correctness of the masker output. Thus, we are assuming the following:

Hypothesis 2: The events S and $(d_*=1)$ are each independent of the event $(d_{i(m)}=d_*)$.

The following theorem gives bounds on the performance measures of the detector when implemented with an error masker.

Theorem 2: If Hypothesis 2 is satisfied, then

$$P_{D,I}(\alpha) \geq P(d_{i(m)}=d_*)P_{D*}$$

$$P_{FA,I}(\alpha) \leq 1 - P(d_{i(m)}=d_*)(1 - P_{FA*}).$$

The bounds given above are not very useful unless we can evaluate $P(d_{i(m)}=d_*)$. As a first step, we shall make the following assumption concerning the error masker.

Hypothesis 3: The error masker is never faulty.

Let $G(\alpha,\gamma)$ be the probability that at most γ device outputs are not equal to d_*. It is clear that under Hypothesis 3, if α is odd, the output $d_{i(m)}$ of the masker is equal to d_* if and only if at most $\frac{\alpha-1}{2}$ device outputs are not equal to d_*; and if α is even, $d_{i(m)}$ is equal to d_* if at most $\frac{\alpha-2}{2}$, and only if at most $\frac{\alpha}{2}$, device outputs are not equal to d_*. Thus, the following lemma has been proved.

Lemma 1: If Hypothesis 3 is satisfied, then

$$P(d_{i(m)}=d_*) = G(\alpha,\tfrac{\alpha-1}{2}),\ \alpha \text{ odd}$$

$$G(\alpha,\tfrac{\alpha}{2}) \geq P(d_{i(m)}=d_*) \geq G(\alpha,\tfrac{\alpha-2}{2}),\ \alpha \text{ even}.$$

In order to explicitly relate the performance bounds to hardware reliability, we shall make the following assumption.

Hypothesis 4:

(i) Each computing device has the same probability q of being non-faulty

(ii) The computing devices fail independently.

Let $H(\alpha,\gamma)$ be the probability that at most γ devices are faulty. Hypothesis 4 implies

$$H(\alpha,\gamma) = \sum_{j=0}^{\gamma} \frac{\alpha!}{j!(\alpha-j)!}(1-q)^j q^{(\alpha-j)}. \tag{3}$$

It is clear that if at most γ devices are faulty, then at most γ device outputs will not equal d_*, and thus

$$G(\alpha,\gamma) \geq H(\alpha,\gamma). \tag{4}$$

In view of the preceding discussion, the following corollary is immediately obtained.

Corollary 2: If Hypotheses 2, 3 and 4 are satisfied, then

$$P_{D,I}(\alpha) \geq R_I(\alpha)P_{D*} \tag{5}$$

$$P_{FA,I}(\alpha) \leq 1 - R_I(\alpha)(1 - P_{FA*}) \tag{6}$$

where

$$R_I(\alpha) = H(\alpha, \tfrac{\alpha-1}{2}),\ \alpha \text{ odd} \tag{7}$$

$$R_I(\alpha) = H(\alpha, \tfrac{\alpha-2}{2}),\ \alpha \text{ even} \tag{8}$$

and where $H(.,.)$ is given by (3).

IV. PERFORMANCE USING ERROR DETECTION AND MASKING

Another way to possibly minimize detector performance degradation is to use both error detection and error masking. To implement such a scheme we replicate the original computing device $\alpha-1$ times, send the original data sample to each device, and then send all the device outputs $d_1, d_2, \ldots, d_\alpha$ to both an error masker and an error detector. The error masker operates as described in Section III in choosing an index $i(m)$ and producing an output $d_{i(m)}$. The error detector compares the d_i's and produces a boolean variable b that equals zero if at least ξ d_i's are identical, and that equals one otherwise. If $b = 0$, we make a decision concerning the presence or absence of the signal using the decision variable $d_{i(m)}$. If $b = 1$, we make no decision concerning the signal.

In this case, the performance measures of the signal detector are the probability of detection $P_{D,II}(\alpha,\xi)$ and the probability of false alarm $P_{FA,II}(\alpha,\xi)$, given that we in fact make a decision concerning the signal, and the probability $P_{FR,II}(\alpha,\xi)$ that we do not make a decision

concerning the signal even though the output of the masker is equal to d_* (probability of false rejection). These quantities are defined as follows:

$$P_{D,II}(\alpha,\xi) = P(d_{i(m)}=1 \,|\, S \text{ and } b=0)$$

$$P_{FA,II}(\alpha,\xi) = P(d_{i(m)}=1 \,|\, S^c \text{ and } b=0)$$

$$P_{FR,II}(\alpha,\xi) = P(b=1 \,|\, d_{i(m)}=d_*).$$

In order to obtain bounds on these performance measures, we shall assume, in addition to Hypothesis 2, that the presence of the signal and the value of the theoretical decision variable are each independent of the number of identical device outputs. Thus, we are assuming:

Hypothesis 5: The events S and $(d_*=1)$ are each independent of the event $(b=0)$.

The following theorem gives bounds on the probabilities of detection and false alarm for a signal detector implemented with an error detector and masker.

Theorem 3: If Hypotheses 2 and 5 are satisfied, then

$$P_{D,II}(\alpha,\xi) \geq P(d_{i(m)}=d_* \,|\, b=0)P_{D*}$$

$$P_{FA,II}(\alpha,\xi) \leq 1 - P(d_{i(m)}=d_* \,|\, b=0)(1 - P_{FA*}).$$

As a first step toward evaluating the quantity $P(d_{i(m)}=d_* \mid b=0)$, we shall make the following assumption concerning the error detector.

Hypothesis 6: The error detector is never faulty.

Note that under Hypothesis 6, if $\xi < \left\lceil \frac{\alpha+2}{2} \right\rceil$ the error detector always produces b = 0 and thus the error detection and masking scheme reduces to just the error masking scheme of the previous section. Therefore, we shall always take $\xi \geq \left\lceil \frac{\alpha+2}{2} \right\rceil$. We can now express $P(d_{i(m)}=d_* \mid b=0)$ as a function of G(.,.).

Lemma 2: If Hypotheses 3 and 6 are satisfied, then for $\xi \geq \left\lceil \frac{\alpha+2}{2} \right\rceil$

$$P(d_{i(m)}=d_* \mid b=0) = \frac{G(\alpha,\alpha-\xi)}{1 + G(\alpha,\alpha-\xi) - G(\alpha,\xi-1)}.$$

The next step in evaluating $P(d_{i(m)}=d_* \mid b=0)$ consists in replacing G(.,.) with H(.,.). Note that Equation (4) implies

$$\frac{G(\alpha,\alpha-\xi)}{1 + G(\alpha,\alpha-\xi) - G(\alpha,\xi-1)} \geq \frac{H(\alpha,\alpha-\xi)}{1 + H(\alpha,\alpha-\xi) - H(\alpha,\xi-1)}$$

which immediately leads to the following corollary.

Corollary 3: If Hypotheses 2, 3, 4, 5 and 6 are satisfied, then for $\xi \geq \left\lceil \frac{\alpha+2}{2} \right\rceil$

$$P_{D,II}(\alpha,\xi) \geq R_{II}(\alpha,\xi) P_{D*} \tag{9}$$

$$P_{FA,II}(\alpha,\xi) \leq 1 - R_{II}(\alpha,\xi)(1 - P_{FA*}) \tag{10}$$

where

$$R_{II}(\alpha,\xi) = \frac{H(\alpha,\alpha-\xi)}{1 + H(\alpha,\alpha-\xi) - H(\alpha,\xi-1)} \tag{11}$$

and where $H(.,.)$ is given by (3).

The quantity $P_{FR,II}(\alpha,\xi)$ is a measure of the price that must be paid when using the error detection and masking scheme. We can obtain a preliminary evaluation of this quantity as follows:

Lemma 3: If Hypotheses 3 and 6 are satisfied, then for $\xi \geq \left\lceil \frac{\alpha+2}{2} \right\rceil$

$$P_{FR,II}(\alpha,\xi) = 1 - \frac{G(\alpha,\alpha-\xi)}{G(\alpha,(\alpha-1)/2)} \quad , \ \alpha \text{ odd}$$

$$P_{FR,II}(\alpha,\xi) \leq 1 - \frac{G(\alpha,\alpha-\xi)}{G(\alpha,\alpha/2)}, \ \alpha \text{ even.}$$

In order to relate the results given above to the hardware reliability, we need the following hypothesis.

Hypothesis 7:

(i) $P(d_i=d_*) = \tilde{q}$, $i = 1, 2, \ldots, \alpha$

(ii) The events $(d_i=d_*)$, $i = 1, 2, \ldots, \alpha$, are mutually independent.

Hypothesis 7 implies

$$G(\alpha,\gamma) = \sum_{j=0}^{\gamma} \frac{\alpha!}{j!(\alpha-j)!}(1-\tilde{q})^j \tilde{q}^{(\alpha-j)}. \tag{12}$$

The next lemma provides a relation between $G(.,.)$ and $H(.,.)$.

Lemma 4: If Hypotheses 4 and 7 are satisfied, then for $\xi \geq \left\lceil \frac{\alpha+2}{2} \right\rceil$

$$\frac{G(\alpha,\alpha-\xi)}{H(\alpha,\alpha-\xi)} \geq \frac{G(\alpha,(\alpha-1)/2)}{H(\alpha,(\alpha-1)/2)}, \ \alpha \text{ odd}$$

$$\frac{G(\alpha,\alpha-\xi)}{H(\alpha,\alpha-\xi)} \geq \frac{G(\alpha,\alpha/2)}{H(\alpha,\alpha/2)}, \quad \alpha \text{ even.}$$

The following corollary completes our task by providing a computable upper bound on $P_{FR,II}(\alpha,\xi)$.

Corollary 4: If Hypotheses 3, 4, 6 and 7 are satisfied, then for $\xi \geq \left\lceil \frac{\alpha+2}{2} \right\rceil$

$$P_{FR,II}(\alpha,\xi) \leq 1 - \frac{H(\alpha,\alpha-\xi)}{H(\alpha,(\alpha-1)/2)}, \quad \alpha \text{ odd} \tag{13}$$

$$P_{FR,II}(\alpha,\xi) \leq 1 - \frac{H(\alpha,\alpha-\xi)}{H(\alpha,\alpha/2)}, \quad \alpha \text{ even} \tag{14}$$

where $H(.,.)$ is given by (3).

V. CONCLUSION

We have analyzed three types of detector implementations. The first does not use any redundancy; the second uses strict redundancy for error masking; and the third uses strict redundancy for both error detection and masking. In the first two cases, we always make a decision concerning the presence of the signal, but in the third case, we sometimes make no decision concerning the signal. Thus, in addition to the probabilities of detection and false alarm, which characterize the first two cases, the third case is also characterized by the probability of false rejection. In this part of our work, we derived the appropriate bounds on these figures of merit using only physically reasonable assumptions. The task of exploiting those bounds for the design of

efficient detector implementations will be carried out in a subsequent paper.

REFERENCES

[1] Helstrom, C. W., Statistical Theory of Signal Detection, Oxford: Pergamon Press, 1968.

[2] Avizienis, A., Fault Tolerance: The Survival Attribute of Digital Systems, Proc. IEEE, Vol. 66, 1978, pp. 1109-1125.

APPENDIX

Proof of Theorem 1:

Let B_1 be the event $(d_1=d_*)$ and let B_1^c be its complement. The definition of P_D implies

$$P_D = \frac{P(d_1=1 \text{ and } S)}{P(S)} = \frac{X + Y}{P(S)}$$

where

$$X = P(d_1=1 \text{ and } S \text{ and } B_1)$$

$$Y = P(d_1=1 \text{ and } S \text{ and } B_1^c).$$

The worst case for the probability of detection occurs when

$$P(d_1=1 \mid S \text{ and } B_1^c) = 0$$

which implies that $Y = 0$. Since the event $(d_1=1 \text{ and } B_1)$ is the same as the event $(d_*=1 \text{ and } B_1)$,

$$X = P(d_*=1 \text{ and } S \text{ and } B_1).$$

Then, using Hypothesis 1,

$$X = P(d_*=1 \text{ and } S)P(B_1)$$

$$= P(d_*=1 \mid S)P(S)P(B_1)$$

and the first part of the theorem follows.

The definition of F_{FA} implies

$$P_{FA} = \frac{P(d_1=1 \text{ and } S^c)}{P(S^c)} = \frac{X + Y}{P(S^c)}$$

where

$$X = P(d_1=1 \text{ and } S^c \text{ and } B_1)$$

$$Y = P(d_1=1 \text{ and } S^c \text{ and } B_1^c).$$

The worst case for the probability of false alarm occurs when

$$P(d_1=1 \mid S^c \text{ and } B_1^c) = 1$$

and therefore, using Hypothesis 1,

$$Y \leq P(S^c \text{ and } B_1^c) = P(S^c)(1 - P(B_1)).$$

The fact that the events $(d_1=1 \text{ and } B_1)$ and $(d_*=1 \text{ and } B_1)$ are identical implies

$$X = P(d_*=1 \text{ and } S^c \text{ and } B_1)$$

and then, using Hypothesis 1,

$$X = P(d_*=1 \text{ and } S^c)P(B_1)$$

$$= P(d_*=1 \mid S^c)P(S^c)P(B_1)$$

and the second part of the theorem follows.

Proof of Theorem 2:

Let $B_{i(m)}$ be the event $(d_{i(m)}=d_*)$ and let $B^c_{i(m)}$ be its complement. The definition of $P_{D,I}(\alpha)$ implies

$$P_{D,I}(\alpha) = \frac{P(d_{i(m)}=1 \text{ and } S)}{P(S)} = \frac{X + Y}{P(S)}$$

where

$$X = P(d_{i(m)}=1 \text{ and } S \text{ and } B_{i(m)})$$

$$Y = P(d_{i(m)}=1 \text{ and } S \text{ and } B^c_{i(m)}).$$

The worst case for the probability of detection occurs when

$$P(d_{i(m)}=1 \mid S \text{ and } B^c_{i(m)}) = 0$$

and therefore $Y = 0$. Since the events $(d_{i(m)}=1 \text{ and } B_{i(m)})$ and $(d_*=1 \text{ and } B_{i(m)})$ are identical,

$$X = P(d_*=1 \text{ and } S \text{ and } B_{i(m)}).$$

Then, using Hypothesis 2,

$$X = P(d_*=1 \text{ and } S)P(B_{i(m)})$$

$$= P(d_*=1 \mid S)P(S)P(B_{i(m)})$$

and the first part of the theorem follows.

The definition of $P_{FA,I}(\alpha)$ implies

$$P_{FA,I}(\alpha) = \frac{P(d_{i(m)}=1 \text{ and } S^c)}{P(S^c)} = \frac{X + Y}{P(S^c)}$$

where

$$X = P(d_{i(m)}=1 \text{ and } S^c \text{ and } B_{i(m)})$$

$$Y = P(d_{i(m)}=1 \text{ and } S^c \text{ and } B^c_{i(m)}).$$

The worst case for the probability of false alarm occurs when

$$P(d_{i(m)}=1 \mid S^c \text{ and } B^c_{i(m)}) = 1$$

and therefore, using Hypothesis 2,

$$Y \leq P(S^c \text{ and } B^c_{i(m)}) = P(S^c)(1 - P(B_{i(m)})).$$

Since the events $(d_{i(m)}=1 \text{ and } B_{i(m)})$ and $(d_*=1 \text{ and } B_{i(m)})$ are identical,

$$X = P(d_*=1 \text{ and } S^c \text{ and } B_{i(m)})$$

and then, using Hypothesis 2,

$$X = P(d_*=1 \text{ and } S^c)P(B_{i(m)})$$

$$= P(d_*=1 \mid S^c)P(S^c)P(B_{i(m)})$$

and the second part of the theorem follows.

Proof of Theorem 3:

The definition of $P_{D,II}(\alpha,\xi)$ implies

$$P_{D,II}(\alpha,\xi) = \frac{P(d_{i(m)}=1 \text{ and } S \text{ and } b=0)}{P(S \text{ and } b=0)} = \frac{X + Y}{P(S \text{ and } b=0)}$$

where

$$X = P(d_{i(m)}=1 \text{ and } S \text{ and } b=0 \text{ and } B_{i(m)})$$

$$Y = P(d_{i(m)}=1 \text{ and } S \text{ and } b=0 \text{ and } B^c_{i(m)}).$$

The worst case for the probability of detection occurs when

$$P(d_{i(m)}=1 \mid S \text{ and } b=0 \text{ and } B^c_{i(m)}) = 0$$

and therefore $Y = 0$. Since the events $(d_{i(m)}=1 \text{ and } B_{i(m)})$ and $(d_*=1 \text{ and } B_{i(m)})$ are identical,

$$X = P(d_*=1 \text{ and } S \text{ and } b=0 \text{ and } B_{i(m)}).$$

Then, using Hypotheses 2 and 5,

$$X = P(d_*=1 \text{ and } S)P(b=0 \text{ and } B_{i(m)})$$

$$= P(d_*=1 \mid S)P(S)P(b=0 \text{ and } B_{i(m)})$$

and the first part of the theorem follows.

The definition of $P_{FA,II}(\alpha,\xi)$ implies

$$P_{FA,II}(\alpha,\xi) = \frac{P(d_{i(m)}=1 \text{ and } S^c \text{ and } b=0)}{P(S^c \text{ and } b=0)} = \frac{X + Y}{P(S^c \text{ and } b=0)}$$

where

$$X = P(d_{i(m)}=1 \text{ and } S^c \text{ and } b=0 \text{ and } B_{i(m)})$$

$$Y = P(d_{i(m)}=1 \text{ and } S^c \text{ and } b=0 \text{ and } B^c_{i(m)}).$$

The worst case for the probability of false alarm occurs when

$$P(d_{i(m)}=1 \mid S^c \text{ and } b=0 \text{ and } B^c_{i(m)}) = 1$$

and therefore, using Hypotheses 2 and 5,

$$Y \leq P(S^c \text{ and } b=0 \text{ and } B^c_{i(m)}) = P(S^c)P(b=0 \text{ and } B^c_{i(m)})$$

Since the events $(d_{i(m)}=1 \text{ and } B_{i(m)})$ and $(d_*=1 \text{ and } B_{i(m)})$ are identical,

$$X = P(d_*=1 \text{ and } S^c \text{ and } b=0 \text{ and } B_{i(m)})$$

and then, using Hypotheses 2 and 5,

$$X = P(d_*=1 \text{ and } S^c)P(b=0 \text{ and } B_{i(m)})$$

$$= P(d_*=1 \mid S^c)P(S^c)P(b=0 \text{ and } B_{i(m)})$$

and the second part of the theorem follows.

Proof of Lemma 2:

If $\xi \geq \left\lceil \frac{\alpha+2}{2} \right\rceil$, the events $(d_{i(m)}=d_* \text{ and } b=0)$, (at least ξ device outputs equal d_*) and (at most $\alpha-\xi$ device outputs do not equal d_*) are equivalent. Also, the event $(b=0)$ is equal to the union of the two disjoint events (at least ξ device outputs equal d_*) and (at least ξ device outputs do not equal d_*). It follows that

$$P(d_{i(m)}=d_* \mid b=0) = \frac{P(d_{i(m)}=d_* \text{ and } b=0)}{P(b=0)}$$

$$= \frac{G(\alpha,\alpha-\xi)}{1 + G(\alpha,\alpha-\xi) - G(\alpha,\xi-1)}$$

and the lemma is proved.

Proof of Lemma 3:

The definition of $P_{FR,II}(\alpha,\xi)$ implies

$$P_{FR,II}(\alpha,\xi) = 1 - P(b=0 \mid d_{i(m)}=d_*)$$

$$= 1 - \frac{P(b=0 \text{ and } d_{i(m)}=d_*)}{P(d_{i(m)}=d_*)}$$

As in the proof of Lemma 2, if $\xi \geq \left\lceil \frac{\alpha+2}{2} \right\rceil$, then

$$P(d_{i(m)}=d_* \text{ and } b=0) = G(\alpha,\alpha-\xi).$$

The proof is completed using Lemma 1.

Proof of Lemma 4:

Let α be odd and $\xi \geq \frac{\alpha+3}{2}$. Then, letting $w = (1-q)/q$ and $\tilde{w} = (1-\tilde{q})/\tilde{q}$, Equations (3) and (12) imply

$$G(\alpha,(\alpha-1)/2) = G(\alpha,\alpha-\xi) + \tilde{q}^{\alpha} \sum_{j=\alpha-\xi+1}^{(\alpha-1)/2} \frac{\alpha!}{j!(\alpha-j)!} \tilde{w}^j$$

$$H(\alpha,(\alpha-1)/2) = H(\alpha,\alpha-\xi) + q^{\alpha} \sum_{j=\alpha-\xi+1}^{(\alpha-1)/2} \frac{\alpha!}{j!(\alpha-j)!} w^j.$$

The desired result will be true if and only if

$$G(\alpha,\alpha-\xi) q^{\alpha} \sum_{j=\alpha-\xi+1}^{(\alpha-1)/2} \frac{\alpha!}{j!(\alpha-j)!} w^j \geq H(\alpha,\alpha-\xi) \tilde{q}^{\alpha} \sum_{j=\alpha-\xi+1}^{(\alpha-1)/2} \frac{\alpha!}{j!(\alpha-j)!} \tilde{w}^j$$

or, equivalently, if and only if

$$\sum_{i=0}^{\alpha-\xi} \sum_{j=\alpha-\xi+1}^{(\alpha-1)/2} \frac{\alpha!}{i!(\alpha-i)!} \frac{\alpha!}{j!(\alpha-j)!} (\tilde{w}^i w^j - \tilde{w}^j w^i) \geq 0.$$

Now, Equation (4) implies that $w \succeq \tilde{w}$, and since $j > i$, the above inequality holds and the first part of the lemma follows. The proof of the second part of the lemma is similar to the above.